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Hamiltonian Formulation and Action Principle for the Lorentz-Dirac System

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The possibility of constructing a Lagrangian and Hamiltonian formulation is examined for a radiating point-like charge usually described by the classical Lorentz-Dirac equation. It turns out that the latter equation cannot be obtained from the variational principle, and, furthermore, has nonphysical solutions. It is proposed to consider a physically equivalent set of reduced equations which admit a Hamiltonian formulation with non-canonical Poisson brackets. As an example, the effective dynamics of a non-relativistic particle moving in a homogeneous magnetic field is considered. The proposed Hamiltonian formulation may be considered as a first step to a consistent quantization of the Lorentz-Dirac system.

KEY WORDS: Lorentz-Dirac equation; systems with higher derivatives

1. INTRODUCTION

A remarkable peculiarity of the classical electrodynamics of point-like particles is the possibility of a consistent analysis of the radiation damping force in the framework of local equations of motion which do not contain dynamical fields. These equations are known as the Lorentz-Dirac equations and differ from the usual Lorentz equations by additional terms with a third-order derivative in the trajectory time (Dirac, 1938; Landau and Lifshitz, 1962; Poisson, 2006). The presence of odd higher-order time derivative implies two immediate consequences. First of all, the dynamics of a particle is no longer reversible in time, which is an agreement with the intuitive concept of the physical irreversibility of a radiation process. In the second place, as distinct from the ordinary classical mechanics, in order to assign the initial data one has to indicate not only the initial position and velocity of a particle but also the initial acceleration. The latter circumstance leads to the fact that, together with the physically sensible solutions, the equations contain numerous solutions which do not admit a reasonable physical interpretation

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(for example, self-accelerating solutions (Landau and Lifshitz, 1962; Poisson, 2006). Furthermore, in contrast to the Lorentz equation, the Lorentz-Dirac equation is not a Lagrangian equation, i.e., it does not follow from the variational principle, which makes impossible a direct quantization of this system.

It should be noted that, on both the quantum and classical level, Maxwell's electrodynamics is an essentially perturbative theory, where calculations are made only in the form of asymptotic series in the coupling constant (the electric charge e). In the general case, these series are divergent; however, the calculation of merely the first terms of an expansion series usually provides a good approximation to the experimental data, in view of the smallness of the fine structure constant. At the classical level, a consistent perturbative treatment of electrodynamics leads to the following selection rule: the physical solutions of the Lorentz-Dirac equation are only those which have a smooth limit of vanishing interaction, i.e., when the particle charge tends to zero. It turns out that all such solutions can be described by one second-order equation, the so-called reduced Lorentz-Dirac equation.

In this paper, we investigate the possibility of constructing a Lagrangian and Hamiltonian formulation for the classical and reduced Lorentz-Dirac equations in the framework of a generalized inverse problem of the calculus of variations. The construction of the Hamiltonian description of a radiating charge can provide a basis for a quantum-mechanical description of the effects of radiation damping, alternative to the quantum field description in the same sense in which the Lorentz-Dirac equation allows one to consider radiation reaction in classical electrodynamics.

The article is organized as follows. In Section 2, we prove that the classical Lorentz-Dirac equation, as well as its nonrelativistic analogue, does not allow the existence of an integrating multiplier (Havas, 1957, 1973; Dodonov *et al.*, 1978), and, therefore, cannot be obtained from the variational principle for a non-local action functional. Section 3 is devoted to the physically equivalent set of reduced equations. Firstly, we discuss the procedure of a perturbative reduction of order for the classical Lorentz-Dirac equation. The aim of this procedure is to derive such a second-order equation that would be equivalent to the initial Lorentz-Dirac equation in the sector of physical solutions and would have no other (nonphysical) solutions. Next, we discuss the inverse problem of the calculus of variations for the reduced nonrelativistic Lorentz-Dirac equation. As an example, we consider the problem of self-consistent dynamics for a charge in a homogeneous magnetic field (the synchrotron problem). It is shown that this system does not have a satisfactory Lagrangian description: none of the possible Lagrangians yields, in the limit of vanishing interaction, the standard Lagrangian of a free particle. Nevertheless, as shown in Section 4, a satisfactory description can be found in a Hamiltonian formalism. The presence of radiation damping leads, however, to the fact that the Poisson brackets are non-canonical, as well as to the absence of phase-space polarization (Woodhouse, 1992), i.e., there is no separation of the variables into position coordinates and conjugate momenta, which prevents a transition to the second-order Lagrangian formalism.

2. IMPOSSIBILITY OF A LAGRANGIAN DESCRIPTION OF THE CLASSICAL LORENTZ-DIRAC SYSTEM

Consider the question of the existence of an action functional for the relativistic Lorentz–Dirac equation (Landau and Lifshitz, 1962; Poisson, 2006)

$$g_\mu = -m\ddot{x}_\mu + \frac{e}{c}F_{\mu\nu}\dot{x}^\nu + \frac{2e^2}{3c^3}\left(\dddot{x}_\mu - \frac{1}{c^2}\dot{x}_\mu\ddot{x}_\nu\ddot{x}^\nu\right) = 0, \tag{1}$$

describing the effective dynamics of a point-like charge with radiation back-reaction. Here, $x^\mu = (t, \mathbf{x})$ are the coordinates of the particle in a four-dimensional space-time; $F_{\mu\nu} = (\mathbf{E}, \mathbf{H})$ is the electromagnetic field tensor; the constants c and e are the speed of light and the electric charge, respectively.

As was already mentioned, this equation is not Lagrangian, i.e., it cannot be obtained by taking a variation of some action functional $S[x]$. However, in a more general setting of this problem there arises the question of the existence of a non-singular matrix $h^\mu_\nu(t, x, \dot{x}\ldots)$ such that for the equivalent set of equations $\acute{g}_\nu = h^\mu_\nu g_\mu = 0$ the inverse problem of the calculus of variations already has a solution, or, equivalently, there does exist an action functional for this system. The matrix h^μ_ν is called an integrating multiplier[2] (Havas, 1957, 1973; Dodonov *et al.*, 1978). In what follows, by the inverse problem of the calculus of variations we understand the finding of an integrating multiplier that converts a given system of differential equations into a total variational derivative. Even in this (extended) setting, it is known that the inverse problem of the calculus of variations is not always solvable, and even if it does have a solution, this solution is not unique (Dodonov *et al.*, 1978; Henneaux, 1982; Morandi *et al.*, 1990; Anderson and Thompson, 1982; Hojman and Urrutia, 1981; Douglas, 1941).

Let there exist an integrating multiplier h^ν_μ for Eq. (1), i.e., there exists an action functional $S[x]$ whose variation yields the system

$$h^\nu_\mu g_\nu = 0 \tag{2}$$

Since Eq. (1) do not include derivatives higher than the third order, the corresponding Lagrangian depends only on the first- and second-order derivatives of the trajectory; moreover, the dependence on the second-order derivatives must be linear. This implies

$$L = a(x, \dot{x}) + b_\mu(x, \dot{x})\ddot{x}^\mu\,. \tag{3}$$

[2] Not to be confused with a similar notion in the theory of ordinary differential equations.

Taking a variation of the action $S[x]$ and making a comparison between the result and Eq. (2), we conclude that

$$\frac{\partial b_\mu}{\partial \dot{x}^\nu} - \frac{\partial b_\nu}{\partial \dot{x}^\mu} = \gamma h_{\mu\nu} ,$$
$$c^2 \left(\frac{\partial^2 b_\nu}{\partial \dot{x}^\mu \partial \dot{x}^\lambda} - \frac{\partial^2 b_\mu}{\partial \dot{x}^\nu \partial \dot{x}^\lambda} \right) \ddot{x}^\nu \ddot{x}^\lambda = -\gamma h_{\mu\lambda} \dot{x}^\lambda \ddot{x}_\nu \ddot{x}^\nu , \tag{4}$$

where $\gamma = 2e^2/3c^3$. In particular, it is evident that the matrix $h_{\mu\nu} = h_\mu^\lambda \eta_{\lambda\nu}$ ($\eta_{\mu\nu}$ is the Minkowsky metrics) must be antisymmetric. Differentiating the first of Eq. (4) w.r.t. $\dot{x}^\lambda$, and substituting the result into the second equation, we obtain

$$\frac{\partial h_{\mu\sigma}}{\partial \dot{x}^\lambda} \ddot{x}^\sigma \ddot{x}^\lambda = -\frac{1}{c^2} h_{\mu\alpha} \dot{x}^\alpha \ddot{x}_\nu \ddot{x}^\nu . \tag{5}$$

As long as $h_{\mu\nu}$ does not depend on $\ddot{x}^\mu$, the above equation yields

$$\frac{\partial h_{\mu\nu}}{\partial \dot{x}^\lambda} = -\frac{1}{c^2} h_{\mu\alpha} \dot{x}^\alpha \eta_{\nu\lambda} . \tag{6}$$

Since $h_{\mu\nu}$ is antisymmetric, we arrive at the relation

$$h_{\mu\alpha} \dot{x}^\alpha \eta_{\nu\lambda} = -h_{\nu\alpha} \dot{x}^\alpha \eta_{\mu\lambda} ,$$

whose contraction with $\eta^{\nu\lambda}$ leads to

$$h_{\mu\nu} \dot{x}^\nu = 0 . \tag{7}$$

Then (6) implies

$$\frac{\partial h_{\mu\nu}}{\partial \dot{x}^\lambda} = 0,$$

which means that $h_{\mu\nu}$ does not depend on $\dot{x}^\lambda$, and then, due to (7), we conclude that the matrix $h_{\mu\nu}$ is degenerate. Thus, we have arrived at a contradiction. Classical Lorentz-Dirac Eq. (1) does not admit the existence of an integrating multiplier.

For the non-relativistic Lorentz–Dirac equation

$$\mathbf{g} = -m\ddot{x} + e\mathbf{E} + \frac{e}{c}[\dot{x}, \mathbf{H}] + \frac{2e^2}{3c^3} \dddot{\mathbf{x}} = 0, \tag{8}$$

the same arguments as above show that the matrix of an integrating multiplier should be antisymmetric, and this, again, contradicts the condition of nonsingularity, since the system (8) is a set of three equations, and hence an integrating multiplier must be a third-rank matrix; however, any antisymmetric matrix of an odd rank is necessarily degenerated.

The presence of an additional item with a third-order time derivative in the Lorentz–Dirac equation, apart from the problem of the variational principle, leads to a certain difficulty related to a physical interpretation. Firstly, in accordance with the postulates of classical mechanics, a state of any mechanical system must be uniquely determined by assigning its position and velocity, which are obviously insufficient for assigning initial data to third-order equations. Secondly, as shown by a simple analysis, together with physically sensible solutions, Eqs. (1) and (8) admit a set of nonphysical (e.g., self-accelerating) solutions (Landau and Lifshitz, 1962; Poisson, 2006).

Both mentioned problems are closely related to each other and have a common solution. Namely, *it is postulated that only those solutions of the Lorentz–Dirac equation are physical trajectories that have a smooth limit as the charge of a particle goes to zero*. This requirement turns out to actually eliminate the pathological solutions, and, furthermore, each physical trajectory can be uniquely determined by assigning the initial position and velocity of a particle. The meaning of the imposed condition becomes obvious if one observes that the third-order time derivative enters the Lorentz–Dirac equation being multiplied by the perturbation e^2. Therefore, in the limit of vanishing interaction ($e \to 0$) the order of the Lorentz–Dirac equation effectively reduces to two, after which this equation describes the ordinary free-particle motion. In other words, within the framework of a perturbative treatment of the electromagnetic interaction, when the coupling constant e is regarded as small, the presence of a term with a third-order time derivative is treated not as the appearance of additional degrees of freedom of the particle (which would be absurd from the physical point of view) but merely as a small deformation of the free particle dynamics.[3]

Instead of extracting smooth (in e) solutions of a third-order equation with a perturbation at the higher derivative, it is possible to set the problem of finding such a second-order equation whose solutions should obey the initial Lorentz–Dirac equation and be automatically smooth. The procedure of constructing such a reduced equation is called the *perturbative reduction of order*.

3. PHYSICALLY EQUIVALENT SET OF REDUCED EQUATIONS

3.1. Perturbative Reduction of Order in the Lorentz–Dirac Equation

Let there exist such a second-order equation,

$$\ddot{x}^i = f^i(x, \dot{x}, e) \quad i = 1, 2, 3 \tag{9}$$

that (a) all solutions $x^i(t, e)$ are smooth functions in a neighborhood of $e = 0$, and (b) it obeys the nonrelativistic Lorentz–Dirac equation (8). The fulfillment of condition (a) can be guaranteed after requiring, for instance, that the right-hand side of Eq. (9) be an analytic function of e. Let us examine condition (b).

[3] At the same time, the value of charge e, playing the role of a deformation parameter, may be more than small. All that is required is the existence of a smooth limit in a solution when $e \to 0$.

Differentiating Eq. (9) one time w.r.t. t, we obtain

$$\dddot{x}^i = \frac{\partial f^i}{\partial x^j}\dot{x}^j + \frac{\partial f^i}{\partial \dot{x}^j}\ddot{x}^j .$$

Using Eq. (9) once again, we have

$$\dddot{x}^i = \frac{\partial f^i}{\partial x^j}\dot{x}^j + \frac{\partial f^i}{\partial \dot{x}^j} f^j . \tag{10}$$

Assuming now that all solutions (9) also obey Eq. (8), and then expressing in the latter equation $\dddot{x}^i$ and $\ddot{x}^i$ through $\dot{x}^i$ and x^i, with the help of (9), (10), we obtain an identity. This yields the following equation for the function $f^i(x, \dot{x})$:

$$mf^i = eE^i + \frac{e}{c}[\dot{x}, H]^i + \frac{2e^2}{3c^3}\left(\frac{\partial f^i}{\partial x^j}\dot{x}^j + \frac{\partial f^i}{\partial \dot{x}^j} f^j\right) . \tag{11}$$

A solution of this equation can be found in the form of a power series in e:

$$f^i = \sum_{k=0}^{\infty} e^k f^i_{(k)} \tag{12}$$

where

$$\begin{aligned} f^i_{(0)} &= 0 , \\ f^i_{(1)} &= (m)^{-1}E^i + (mc)^{-1}[\dot{x}, H]^i , \\ f^i_{(k)} &= \tfrac{2}{3mc^3}\left(\tfrac{\partial f^i_{(k-2)}}{\partial x^j}\dot{x}^j + \sum_{l=0}^{k-2}\tfrac{\partial f^i_{(k-2-l)}}{\partial \dot{x}^j} f^j_{(l)}\right) , \quad k \geq 2 . \end{aligned}$$

Thus, we have obtained a second-order equation describing the effective dynamics of a charged particle in an electromagnetic field with a radiation back-reaction force. All solutions of this equation are also solutions of the initial Lorentz–Dirac equation, and, furthermore, they are smooth in e.

For certain simple configurations of external fields, the expression for the force f^i can be found in a simple form. For instance, in the case of a homogeneous magnetic field, $\mathbf{H} = (0, 0, H)$, $\mathbf{E} = 0$, the reduced Lorentz-Dirac equation takes the form

$$\begin{aligned} \ddot{x} &= \alpha\dot{x} - \beta\dot{y} \\ \ddot{y} &= \beta\dot{x} + \alpha\dot{y} \\ \ddot{z} &= 0 , \end{aligned} \tag{13}$$

where, in the system of units $m = c = 1$, we have

$$\alpha = \frac{6 - \sqrt{6}\sqrt{3 + \sqrt{9 + 64e^6H^2}}}{8e^2} \approx -\frac{2}{3}e^4H^2, \quad \beta = \frac{eH\sqrt{6}}{\sqrt{3 + \sqrt{9 + 64e^6H^2}}} \approx eH . \tag{14}$$

If one formally sets $\alpha = 0$ in Eq. (13), they become the usual Lorentz equations describing the dynamics of a charged particle in the "effective" magnetic field $\mathbf{B} = (0, 0, \beta/e)$. In this case, the trajectories are concentric circles. For $\alpha \neq 0$ the particle spirals at the origin of xy-plane. In view of this parameter, α can be identified with the coefficient of radiation friction.

3.2. Construction of an Action Functional for a Set of Second-Order Equations

In this section, we consider the problem of constructing an action functional for a set of ordinary differential equations of the form

$$\ddot{x}^i = f^i(t, x, \dot{x}) \tag{15}$$

In the case of one degree of freedom, the problem was solved by Darboux (1894). The case of two degrees of freedom was investigated by Douglas (1941); in particular, he presented examples of second-order equations which do not admit an integrating multiplier. The general case has been examined by many authors (see, e.g., (Havas, 1957, 1973; Henneaux, 1982; Morandi *et al.*, 1990; Anderson and Thompson, 1982; Hojman and Urrutia, 1981)).

Technically, the question of the existence of an integrating multiplier is reduced to the analysis of compatibility for the set of linear partial differential equations arising from the condition of commutativity of variational derivatives, so called Helmholtz condition (von Helmholtz, 1887). Below, we will present a simple method of deriving these equations, which does not appeal to the theory of generalized functions.

The most general action functional for a set of second-order differential equations has the form $S[x] = \int dt L(x, \dot{x}, t)$. The extremals of this functional are solutions of the corresponding Euler–Lagrange equation

$$\frac{\delta S}{\delta x^i} = \frac{\partial L}{\partial x^i} - \frac{\partial^2 L}{\partial t \partial \dot{x}^i} - \frac{\partial^2 L}{\partial \dot{x}^i \partial x^j}\dot{x}^j - \frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j}\ddot{x}^j = 0\,. \tag{16}$$

Let S be an action functional for the system (15), then all solutions of Eqs. (15) and (16) must coincide. This means that if we express the acceleration $\ddot{x}^j$ from the Euler–Lagrange equations through the force $f^j(x, \dot{x})$, then we obtain the identity

$$\frac{\partial L}{\partial x^i} - \frac{\partial^2 L}{\partial t \partial \dot{x}^i} - \frac{\partial^2 L}{\partial \dot{x}^i \partial x^j}\dot{x}^j - \frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j} f^j(t, x, \dot{x}) = 0\,. \tag{17}$$

For a given set of functions f^i, this yields a set of second-order partial differential equations for L. It is also necessary to demand that the corresponding Hessian matrix should be non-degenerate:

$$h_{ij} = \frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j}\,, \quad \det(h_{ij}) \neq 0\,. \tag{18}$$

The validity of the above condition implies that one can reduce the set of the Euler–Lagrange equations to the usual form (15).

Obviously, the Hessian matrix plays the role of an integrating multiplier to the initial system (15). In the general case, after imposing condition (18) the set of equations for the Lagrange function may be overfull. Therefore, our strategy will consist in using consequences of Eq. (17) in order to obtain a complete set of equations for the Hessian matrix, whose validity will solve the initial set of equations for the Lagrange function.

First of all, it immediately follows from the definition of the matrix h_{ij} that it is symmetric and obeys the identity

$$\frac{\partial h_{ij}}{\partial \dot{x}^k} = \frac{\partial h_{kj}}{\partial \dot{x}^i}\,. \tag{19}$$

Then, taking a derivative of (17) w.r.t. $\dot{x}^k$, we obtain

$$\frac{\partial^2 L}{\partial x^i \partial \dot{x}^k} - \frac{\partial^3 L}{\partial t \partial \dot{x}^i \partial \dot{x}^k} - \frac{\partial^2 L}{\partial \dot{x}^i \partial x^k} - \frac{\partial^3 L}{\partial \dot{x}^i \partial x^j \partial \dot{x}^k}\dot{x}^j - \frac{\partial}{\partial \dot{x}^k}\left(\frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j} f^j\right) = 0\,. \tag{20}$$

The symmetric part of the above relation yields the following equation for h_{ij}:

$$\hat{D} h_{ik} + \frac{1}{2}\left(h_{ij}\frac{\partial f^j}{\partial \dot{x}^k} + h_{kj}\frac{\partial f^j}{\partial \dot{x}^i}\right) = 0\,, \tag{21}$$

where

$$\hat{D} = \frac{\partial}{\partial t} + \dot{x}^j \frac{\partial}{\partial x^j} + f^j \frac{\partial}{\partial \dot{x}^j}\,. \tag{22}$$

Extracting the antisymmetric part, we arrive at the equation

$$2\left(\frac{\partial^2 L}{\partial x^i \partial \dot{x}^k} - \frac{\partial^2 L}{\partial \dot{x}^i \partial x^k}\right) = h_{ij}\frac{\partial f^j}{\partial \dot{x}^k} - h_{kj}\frac{\partial f^j}{\partial \dot{x}^i}\,, \tag{23}$$

whose differentiation w.r.t. $\dot{x}^l$ yields another equation for the Hessian matrix:

$$\begin{aligned} &\frac{\partial h_{kl}}{\partial x^i} - \frac{\partial h_{il}}{\partial x^k} = \frac{1}{2}\frac{\partial}{\partial \dot{x}^l} A_{ik}\,, \\ &A_{ik} = h_{ij}\frac{\partial f^j}{\partial \dot{x}^k} - h_{kj}\frac{\partial f^j}{\partial \dot{x}^i}\,. \end{aligned} \tag{24}$$

Now, differentiating Eq. (17) w.r.t. x^k and extracting the antisymmetric part, we have

$$\left(\frac{\partial}{\partial t} + \dot{x}^j \frac{\partial}{\partial x^j}\right)\left(\frac{\partial^2 L}{\partial \dot{x}^k \partial x^i} - \frac{\partial^2 L}{\partial \dot{x}^i \partial x^k}\right) - \frac{\partial}{\partial x^k}(h_{ij} f^j) + \frac{\partial}{\partial x^i}(h_{kj} f^j) = 0\,.$$

With allowance for Eqs. (21), (23) and (24), the above condition can be rewritten as

$$h_{ij} B_k^j - h_{kj} B_i^j = 0\,, \tag{25}$$

where

$$B_j^i = \frac{1}{2}\frac{\partial f^i}{\partial \dot{x}^m}\frac{\partial f^m}{\partial \dot{x}^j} - \hat{D}\frac{\partial f^i}{\partial \dot{x}^j} + 2\frac{\partial f^i}{\partial x^j}\,. \tag{26}$$

One can verify that Eqs. (19), (21), (24), (25) are the same equations for the integrating multiplier h_{ij} that arise from the commutativity condition for variational derivatives. Furthermore, if there exists (and is known) a Hessian h_{ij}, then the Lagrange function L can be determined from the equation

$$\frac{\partial L}{\partial x^i} - \frac{\partial^2 L}{\partial t \partial \dot{x}^i} - \frac{\partial^2 L}{\partial \dot{x}^i \partial x^j}\dot{x}^j = h_{ij} f^j\,. \tag{27}$$

As an example, let us consider the problem of constructing a Lagrange function for a charged particle in a homogeneous magnetic field with a radiation reaction force (13). Since the motion along the z-axis is decoupled, it is sufficient to examine the system of the first two equations:

$$\begin{aligned} \ddot{x} &= \alpha\dot{x} - \beta\dot{y}\,, \\ \ddot{y} &= \beta\dot{x} + \alpha\dot{y}\,, \end{aligned} \tag{28}$$

In this case,

$$B_k^i = \frac{1}{2}\begin{pmatrix} \alpha^2 - \beta^2 & -2\alpha\beta \\ 2\alpha\beta & \alpha^2 - \beta^2 \end{pmatrix},$$

and condition (25) implies that

$$\mathrm{Tr}(h_{ij}) = h_{11} + h_{22} = 0\,. \tag{29}$$

Next, it is easy to see that the general solution of Eqs. (19), (21), (24) is determined by one arbitrary function $\phi(z, w)$ and has the form

$$h_{ij} = \begin{pmatrix} F + \bar{F} & i(F - \bar{F}) \\ i(F - \bar{F}) & -(F + \bar{F}) \end{pmatrix}. \tag{30}$$

Here, $F = \phi(\dot{\xi}e^{-\gamma t}, \dot{\xi} - \gamma\xi)e^{-\gamma t}, \xi = x + iy, \gamma = \alpha + i\beta$, and the bar stands for complex conjugation.

Setting $\phi = 1/z$, we obtain the simplest time-independent solution:

$$h_{ij} = \frac{2}{\dot{x}^2 + \dot{y}^2}\begin{pmatrix} \dot{x} & \dot{y} \\ \dot{y} & -\dot{x} \end{pmatrix}. \tag{31}$$

Substituting this Hessian into Eq. 27), we find the following Lagrangian:

$$L = \frac{1}{2}\dot{x}\ln(\dot{x}^2 + \dot{y}^2) + \dot{y}\,\mathrm{arctg}\left(\frac{\dot{x}}{\dot{y}}\right) + \alpha x - \beta y\,. \tag{32}$$

The corresponding Euler–Lagrange equations

$$\frac{\ddot{x}\dot{x} + \ddot{y}\dot{y}}{\dot{x}^2 + \dot{y}^2} = \alpha\,, \quad \frac{\ddot{x}\dot{y} - \ddot{y}\dot{x}}{\dot{x}^2 + \dot{y}^2} = \beta\,, \tag{33}$$

are obviously equivalent to the initial Eq. (28) with the exception of the point $\dot{x} = \dot{y} = 0$.

So, formally, the problem of constructing a Lagrangian description for a self-consistent dynamics of a charged particle in a uniform magnetic field does admit a solution. Let us remind, however, that, in accordance with the perturbative treatment of the electromagnetic interaction, we require in the limit $e \to 0$ (equivalently $\alpha \to 0$, $\beta \to 0$) that the Lagrangian L should transform into the Lagrangian of a free particle,

$$L_0 = \frac{1}{2}(\dot{x}^2 + \dot{y}^2)\,, \tag{34}$$

modulo a total derivative. Unfortunately, neither the Lagrangian (32) nor any other Lagrangian constructed from the matrix (30) has a correct free limit. This is because, according to (29), the trace of the Hessian matrix of any Lagrangian for the set of Eq. (28) must be equal to zero, and this property holds true after the limit is taken. On the other hand, the trace of the Hessian matrix of a Lagrangian L_0 is equal to 2. This contradiction proves the statement.

Nevertheless, the absence of a physically satisfactory description of radiation back-reaction in Lagrangian formalism leaves the hope that such a description is possible in the framework of Hamiltonian mechanics (the first-order formalism). Furthermore, an additional argument in favor of the fact that one should be able to solve the inverse problem of the calculus of variations in the first-order formalism is that a Lagrangian action for an arbitrary system of second-order differential equations does not always exist (Douglas, 1941; Dodonov *et al.*, 1978). At the same time, for an equivalent set of first-order differential equations, a Hamiltonian action can always be found (see below a proof of this statement).

4. HAMILTONIAN FORMULATION

The existence of a solution of the inverse problem of the calculus of variations in the first-order formalism (at least, in an everywhere smooth open region of a phase space) is based on the following simple considerations. Assigning a set of $2n$ ordinary differential equations

$$\dot{x}^i = v^i(x, t) \tag{35}$$

is equivalent to selecting a certain vector field $\mathbf{v} = (\mathbf{v}^\mathbf{i})$ in the phase space $\mathbf{R}^{2n}$ of a system. It is known that in a neighborhood of each non-critical point of a vector field it is possible to introduce such coordinates x^i that the vector field has the form $\mathbf{v} = \partial/\partial x^1$. Then a Hamiltonian action for the set of Eq. (35) can be selected as

$$S[x] = \int dt \left(\sum_{k=1}^{n} x^k \dot{x}^{2n-k+1} - x^n \right).$$

It is clear that, after returning to the initial coordinates, this action will not take such a simple form and the corresponding Hamiltonian equations will generally imply non-canonical Poisson brackets. In view of the smoothness of the change of coordinates, a small deformation of the initial vector field implies a small deformation of the Hamiltonian and the Poisson brackets. Therefore, in the first-order formalism it is always possible to ensure a correct limit of vanishing interaction (deformation). A more detailed discussion of these questions can be found, for example, in the works (Havas, 1957, 1973; Hojman and Urrutia, 1981; Santilli, 1977).

However, it should be noted that the construction of a change of coordinates is actually equivalent to the integration of the initial set of equations. Therefore, an explicit solution of the inverse problem of the calculus of variations can be found for a very limited number of systems. Below, we will present a solution of this problem, first of all, for arbitrary system of linear equations, and show that a quadratic action for such system can always be found; however, in the general case, both the Hamiltonian and the Poisson bracket will explicitly depend on time, and thus the theory will be nonstationary. We then consider the case when the functions v^i, that determine the set of Eq. (35), may have an arbitrary dependence on x but do not depend explicitly on time. In this case, a stationary action can be found, but the Poisson bracket and Hamiltonian may be essentially nonlinear. Both these possibilities are illustrated by the examples of a nonrelativistic charged particle in a uniform magnetic field with a radiation back-reaction force (28).

Consider the most general set of first-order linear differential equations with non-constant coefficients,

$$\dot{x}^i = A^i_j(t)x^j + J^i(t), \tag{36}$$

defined on a linear phase space with coordinates x^i. We will always assume the phase space of the system (36) to be even-dimensional, i.e., $i = 1, \ldots, 2n$.

Our first observation is that any such system can be derived from the variation principle for a quadratic action functional if the explicit time-dependence is admitted in the integrand. Consider the following ansatz:

$$S[x] = \frac{1}{2} \int dt (x^i \Omega_{ij}(t)\dot{x}^j - x^i B_{ij}(t)x^j - 2C_i(t)x^i), \tag{37}$$

where

$$\Omega_{ij} = -\Omega_{ji}\,, \quad B_{ij} = B_{ji}\,, \quad \det(\Omega_{ij}) \neq 0\,. \tag{38}$$

Structurally, the functional S is similar to the first-order action associated with the Hamiltonian

$$H = \frac{1}{2}x^i B_{ij}(t)x^j - C_i(t)x^i\,, \tag{39}$$

but, unlike the usual Hamiltonian formalism, we allow the symplectic form Ω to depend on time.

Taking a variation of this action functional, we arrive at the following equations [4]:

$$\frac{\delta S}{\delta x^i} = 0 \Leftrightarrow \dot{x} = \Omega^{-1}\left(B - \frac{1}{2}\dot{\Omega}\right)x + \Omega^{-1}C\,. \tag{40}$$

In order that these equations should be equivalent to the original ones (36), we must set

$$A = \Omega^{-1}\left(B - \frac{1}{2}\dot{\Omega}\right), \quad J = \Omega^{-1}C\,, \tag{41}$$

or, equivalently,

$$\frac{1}{2}\dot{\Omega} = B - \Omega A, \quad C = \Omega J\,. \tag{42}$$

Decomposing the first matrix equation into the symmetric and anti-symmetric parts, we finally obtain

$$\dot{\Omega} = -(\Omega A + A^t\Omega)\,, \quad B = \frac{1}{2}\left(\Omega A - A^t\Omega\right), \quad C = \Omega J\,, \tag{43}$$

with A^t being the transposed matrix A. Only the first relation is nontrivial (it is a linear ODE in Ω), while the other two relations are merely definitions of the matrices B and C.

We remind that the square matrix $\Gamma(t)$ is called the fundamental solution of (36) in case

$$\dot{\Gamma} = A\Gamma\,, \quad \Gamma(0) = 1\,. \tag{44}$$

The columns of this matrix constitute the basis in the linear space of solutions to Eq. (36). Given the matrix Γ, the general solution to the first Eq. (43) can be written as

$$\Omega = \Lambda^t\Omega_0\Lambda\,, \tag{45}$$

[4] Here we use the matrix notation.

where $\Lambda = \Gamma^{-1}$, and $\Omega_0 = -\Omega_0^t$ is a constant non-degenerate matrix. The matrix Ω_0 encodes all the ambiguity in the definition of the quadratic action functional (37) for the given system of ODEs (36).

Let us now construct a quadratic first-order action for the second-order Eq. (28). For that we replace them with an equivalent system of first-order ones. Let us choose the auxiliary variables as

$$p = \dot{x} + \frac{\beta}{2}y, \quad q = \dot{y} - \frac{\beta}{2}x .$$

As a result, we arrive at the equations

$$\begin{aligned} \dot{x} &= p - \frac{\beta}{2}y = v^0 , \\ \dot{p} &= -\frac{\beta}{2}q - \frac{\beta^2}{4}x + \alpha\left(p - \frac{\beta}{2}y\right) = v^1 , \\ \dot{y} &= q + \frac{\beta}{2}x = v^2 , \\ \dot{q} &= \frac{\beta}{2}p - \frac{\beta^2}{4}y + \alpha\left(q + \frac{\beta}{2}x\right) = v^3 . \end{aligned} \tag{46}$$

The quadratic action functional for this system takes the form

$$\begin{aligned} S[x] = \int dt \frac{e^{-\alpha t}}{4(\alpha^2+\beta^2)}&[2a(t)(p\dot{x} - x\dot{p} + q\dot{y} - y\dot{q}) \\ &+ 2b(t)(q\dot{x} - x\dot{q} + y\dot{p} - p\dot{y}) + 2c(t)(p\dot{q} - q\dot{p}) + 2d(t)(x\dot{y} - y\dot{x}) \\ &+ e(t)(p^2+q^2) + f(t)(x^2+y^2) + g(t)(px+qy) + j(t)(qx-py)] , \end{aligned} \tag{47}$$

where

$$a(t) = \alpha^2\cos(\beta t) + \frac{1}{2}\beta^2(e^{-\alpha t} + e^{\alpha t}), \quad b(t) = \alpha^2\sin(Bt) - \alpha\beta e^{\alpha t} + \alpha\beta\cos(\beta t),$$

$$c(t) = e^{-\alpha t}\beta + 2\alpha\sin(\beta t) , \quad d(t) = \frac{1}{4}\beta^3(e^{-\alpha t} - e^{\alpha t}) - \frac{1}{2}\beta^2\alpha\sin(\beta t)$$

$$+\alpha^2\beta(\cos(\beta t) - e^{\alpha t}) , \quad e(t) = e^{-\alpha t}\beta^2 + \alpha^2\cos(\beta t) + \alpha\sin(\beta t)\beta,$$

$$f(t) = \frac{1}{4}\beta\left(\beta^3 e^{-\alpha t} + \beta\alpha^2\cos(\beta t) - \alpha\sin(\beta t)[\beta^2 + 2\alpha^2]\right) ,$$

$$g(t) = -\alpha\cos(\beta t)\beta^2 - \alpha^3\cos(\beta t), \;\; j(t) = -\alpha^3\sin(\beta t) + e^{-\alpha t}\beta^3 + \alpha^2\beta\cos(\beta t).$$

In this example, we can observe an interesting fact. Namely, it can be shown that, the system (46) does not admit a first-order quadratic action involving a stationary symplectic structure and having the standard free limit.

In the case when right-hand side of (35) does not depend manifestly on time, one has to look for an action functional in the form

$$S = \int dt \{A_\mu(x)\dot{x}^\mu - H(x)\}\,. \tag{48}$$

Here, $H(x)$ is the Hamiltonian, and $A_\mu(x)$ is a symplectic potential. Taking a variation of this action, we obtain the following set of equations:

$$\omega_{\mu\nu}\dot{x}^\nu = \frac{\partial H}{\partial x^\mu}\,, \tag{49}$$

where $\omega_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is a symplectic 2-form, which is assumed to be nondegenerate.

If Eq. (49) are equivalent to the initial Eq. (35), then substituting into (49) the vector field v^μ, instead of $\dot{x}^\mu$, one obtains the identity

$$\omega_{\mu\nu}v^\nu = \frac{\partial H}{\partial x^\mu}\,. \tag{50}$$

For a given v^μ, this identity produces a number of conditions for the Hamiltonian H and the symplectic potential A_μ. Contracting the above relation with v^μ, we obtain the equation

$$v^\mu \frac{\partial H}{\partial x^\mu} = 0\,, \tag{51}$$

which implies that the Hamiltonian function H is an integral of motion, and, therefore, it must be conserved by the vector flow v^μ. Besides, by definition, the 2-form $\omega_{\mu\nu}$ must be closed:

$$\partial_\lambda\omega_{\mu\nu} + \partial_\mu\omega_{\nu\lambda} + \partial_\nu\omega_{\lambda\mu} = 0\,. \tag{52}$$

If the symplectic 2-form is known, the symplectic potential can be found from the integral formula

$$A_\mu(x) = \int_0^1 x^\nu \omega_{\nu\mu}(tx)dt + \partial_\mu\varphi(x)\,,$$

where $\varphi(x)$ is an arbitrary function in the phase space.

For a given vector field v^μ, Eqs. (50)–(52) can be solved by the method of characteristics. For instance, in the case of Eq. (46), we obtain

$$H = \frac{1}{2}\left[p^2 + q^2 + \beta(qx - py) + \frac{1}{4}\beta^2(x^2 + y^2)\right]\exp\left\{-\frac{2\alpha}{\beta}\arctan\left[\frac{\bar{p} + c\bar{q}}{c\bar{p} - \bar{q}}\right]\right\}, \tag{53}$$

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & F_3 & -F_2 \\ -E_2 & -F_3 & 0 & F_1 \\ -E_3 & F_2 & -F_1 & 0 \end{pmatrix}, \tag{54}$$

where[5]

$$E_i = \frac{1}{v_0}\left(\epsilon_{ijk} v^j F^k - \frac{\partial H}{\partial x^i}\right),$$

$$F_1 = \frac{k-1}{2}, \quad F_2 = \frac{k+1}{B}, \quad F_3 = \frac{\alpha}{\beta},$$

$$k = -\exp\left\{-\frac{2\alpha}{\beta}\arctan\left[\frac{\bar{p}+c\bar{q}}{c\bar{p}-\bar{q}}\right]\right\},$$

$$\bar{p} = p - \frac{\beta}{2}y, \quad \bar{q} = q + \frac{\beta}{2}x,$$

and c is an arbitrary integration constant.

In the limit of vanishing friction ($\alpha \to 0$), the bracket (54) and Hamiltonian (53) tend to the canonical bracket and to the Hamiltonian of a charged particle in a homogeneous magnetic field, respectively. The same result takes place also in the nonstationary theory.

Notice that the symplectic structure (54), as well as its nonstationary analogue, does not possess the xy-polarization (i.e. the 2-form Ω does not vanish upon a restriction to the 2-plane $x = const$, $y = const$) when $\alpha \neq 0$. This makes impossible an algebraic elimination of p and q from the action, and thus prevents obtaining a second-order action in terms of x and y. The latter fact is in agreement with the statement of the previous section concerning the non-existence of a second-order action functional for Eq. (13) that would pass to the standard action functional for a free particle when $e \to 0$.

Since the inverse problem of the calculus of variations for one and the same set of equations may admit numerous solutions, the quantization problem for the system in question does not have a unique solution. Therefore, it is necessary to impose additional reasonable conditions on the sought-for model, such that the action functional of this model should possess "good" properties. In the case of linear systems, the requirement of the squareness of an action functional could serve as such a condition.

In conclusion, let us note that the presence of a Hamiltonian action provides the possibility of constructing a quantum-mechanical description of the system in question. However, the practical implementation of the quantization procedure

[5] The auxiliary symbols E_i and F_i are not to be confused with the strength of electromagnetic field.

encounters the difficulty that the Poisson brackets are non-canonical. Furthermore, in the phase space of the system the natural polarization (Woodhouse, 1992) is absent. The combination of all these circumstances prevents a direct application of the canonical quantization scheme. Nevertheless, the mentioned difficulties can be overcome by the formalism of deformation quantization (Bayen *et al.*, 1977), which is applicable to the case of non-canonical Poisson brackets and does not require the presence of any polarization.

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